# Order $m\alpha^8$ contributions to the decay rate of Orthopositronium

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**Abstract**

We discuss how contributions to the order $\mathcal{O}(m\alpha^8)$ orthopositronium decay rate can be separated into two categories, one due to relativistic momenta and calculable in terms of QED scattering amplitudes, the other due to low momenta and calculable in the simpler framework of a low-energy effective theory (NRQED). We report new results for all low-momentum contributions, and give a formula relating the remaining contributions to conventional (on-shell) QED scattering amplitudes.

Despite its many successes, quantum electrodynamics has yet to account fully for the decay rate of orthopositronium. The theoretical expression for this rate is [3, 4]

$$\begin{aligned}\lambda_{th} \;=\; & \frac{\alpha^6 mc^2}{\hbar}\frac{(2\pi^2-18)}{9\pi}\Bigg[1-10.282(3)(\frac{\alpha}{\pi}) \\ & +\frac{1}{3}\alpha^2\;\ln\alpha + B\,(\frac{\alpha}{\pi})^2 - \frac{3}{2\pi}\,\alpha^3\,(\ln\alpha)^2\ldots\Bigg]\end{aligned} \tag{1}$$

where the coefficient $B$ has not yet been computed. The measured rate is $\lambda_{exp} = 7.0482(16)\mu\mathrm{s}^{-1}$ [1]. The difference between this value and the known part of $\lambda_{th}$,

$$\lambda_{exp} - \lambda_{th}(B=0) \;=\; 99(16)\times 10^{-4}\mu\mathrm{s}^{-1} \tag{2}$$

is surprisingly large: the coefficient $B$ in $\lambda_{th}$ would have to be about 250 to bring theory and experiment into agreement. Such a large coefficient would be

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unusual, but is by no means impossible, particularly given the big $\mathcal{O}(\alpha/\pi)$ correction. A complete calculation of $B$ is essential before any realistic assessment of the situation is possible.

A calculation of these coefficients using traditional bound-state methods is very complicated. This is because each term in a traditional expansion has contributions from both nonrelativistic and relativistic momenta. This means that approximations only valid for small $p$ or only for large $p$ cannot be readily employed to simplify the analysis. In this paper, we outline a new and simpler procedure for computing $B$. Our analysis is based upon a rigorous nonrelativistic reformulation of QED called nonrelativistic quantum electrodynamics (NRQED) [2]. Using this effective field theory, we are able to separate the $\mathcal{O}(\alpha^2)$ corrections into three parts. Two of these involve soft, nonrelativistic momenta and probe the bound-state nature of the system. The other part involves hard, relativistic momenta and is therefore largely insensitive to the details of binding. We have calculated the nonrelativistic contribution and present the results here. We also show how to extract the relativistic contribution from a calculation of ordinary scattering amplitudes; no bound-state physics is required in this part of the calculation.

Our positronium results also have implications for quarkonium decays. These will be discussed in another paper [8].

The lowest-order decay rate of orthopositronium (O-Ps, $n = J = S = 1$) is given by (we now use natural units, with $c = \hbar = 1$)

$$\begin{aligned}\Gamma_0(\text{O-Ps} \to 3\gamma) &= |\Psi(0)|^2\ \hat{\sigma}_0(0) \\ &= \frac{2\pi^2 - 18}{9\pi}\, m_e \alpha^6 \\ &= 7.2112\,\mu\text{s}^{-1}. \end{aligned} \tag{3}$$

Here $\Psi(0)$ is the ground state Schrödinger-Coulomb wavefunction evaluated at $\vec{r} = 0$, and $\hat{\sigma}_0(p)$ is proportional to the lowest-order decay rate of a *free* electron and positron in an S-state:

$$\hat{\sigma}_0(p) \equiv \frac{1}{4m_e^2}\ \text{Im}\ \mathcal{M}_0^{(l=0)}\left(e\bar{e} \to 3\gamma \to e\bar{e}\right), \tag{4}$$

where $p$ is the magnitude of the electron center-of-mass momentum.

To order $\mathcal{O}(\alpha^2)$, there are three sources of corrections:

1) RADIATIVE CORRECTIONS TO $\hat{\sigma}_0(0)$: These corrections renormalize the lowest-order decay rate:

$$\delta\Gamma_1 \ =\ \delta Z\ \Gamma_0(\text{O-Ps} \to 3\gamma). \tag{5}$$

The "renormalization" constant $\delta Z$ is defined to be the part of the radiative corrections coming from relativistic loop momenta. Thus it is insensitive to the

binding energy of the atom, and can be related to the infrared-finite part of the radiatively corrected annihilation rate $\hat{\sigma}$ for a free electron and positron at $p = 0$:

$$\hat{\sigma}(0) - \hat{\sigma}_{\rm IR}(0) \;=\; (1+\delta Z)\,\hat{\sigma}_0(0), \tag{6}$$

where $\hat{\sigma}_{\rm IR}(0)$ is the infrared part of $\hat{\sigma}(0)$. Using a photon mass to regulate the infrared (threshold) singularities[1]

$$\begin{aligned}\frac{\hat{\sigma}_{\rm IR}(0)}{\hat{\sigma}_0(0)} \;&=\; \frac{m_e\alpha}{\lambda} + (2\ln 2 + 1)\alpha^2\,\frac{m_e^2}{\lambda^2} \\ &-\; 2\times 10.282(3)\frac{\alpha^2}{\pi}\frac{m_e}{\lambda} + \frac{1}{3}\alpha^2\,\ln\frac{\lambda}{m_e} + \mathcal{O}(\alpha^3).\end{aligned} \tag{7}$$

These infrared subtractions remove all contributions due to nonrelativistic loop momenta. Thus $\delta Z$ has a $\lambda$-independent expansion in powers of $\alpha/\pi$:

$$\delta Z \;=\; c_1\frac{\alpha}{\pi} + c_2\left(\frac{\alpha}{\pi}\right)^2 + \ldots. \tag{8}$$

The only $\mathcal{O}(\alpha)$ corrections to the decay rate come from $\delta Z$, and thus, from Eq.[1] $c_1 = -10.282(3)$. The second-order coefficient has not yet been calculated. The large size of $c_1$ suggests that $c_2$ might also be large. Indeed, a small (gauge invariant) subset of the diagrams contributing to $c_2$ has recently been evaluated and found to contribute 28.8(2) to $c_2$ [7].

2) MOMENTUM DEPENDENCE OF $\hat{\sigma}_0(p)$ : Near threshold, the decay rate for a free electron and positron in an S-wave has the form

$$\hat{\sigma}_0(p) \;=\; \hat{\sigma}_0(0) + \delta\hat{\sigma}_0(p) + \mathcal{O}(\frac{p^4}{m_e^4}\hat{\sigma}_0), \tag{9}$$

where $\delta\hat{\sigma}_0(p)$ can be computed to be [6]:

$$\delta\hat{\sigma}_0(p) \;=\; -\,\frac{p^2}{m_e^2}\,\frac{19\pi^2 - 132}{12(\pi^2-9)}\;\hat{\sigma}_0(0). \tag{10}$$

This correction term shifts the decay rate in $\mathcal{O}(\alpha^2)$. Naïvely, one might expect a contribution

$$2\Psi(0)\int\frac{d^3p}{(2\pi)^3}\,\delta\hat{\sigma}_0(p)\,\Psi(p). \tag{11}$$

However, this integral includes contributions already present in the relativistic radiative corrections discussed above (Eq.[5]). This becomes apparent if the Schrödinger equation is used to rewrite the wavefunction:

$$\Psi(p) \;=\; \frac{1}{E_{\rm O\text{-}Ps} - p^2/m_e}\int\frac{d^3q}{(2\pi)^3}\,V_C(\mathbf{p}-\mathbf{q})\,\Psi(q) \tag{12}$$

[1] The precise definition of $\hat{\sigma}_{\rm IR}(0)$ depends upon the regulator used in the NRQED part of the analysis; see refs [2], [5].

where $V_C$ is the Coulomb potential, $V_C(\mathbf{p}-\mathbf{q}) = -e^2/(\mathbf{p}-\mathbf{q})^2$, and $E_{\text{O-Ps}} = -m\alpha^2/4$ is the binding energy. Then our contribution to the orthopositronium decay rate becomes

$$2\,\Psi(0)\int \frac{d^3p}{(2\pi)^3}\,\delta\hat{\sigma}_0(p)\,\Psi(p) \;=\; 2\,\Psi(0)\int \frac{d^3p}{(2\pi)^3}\,\delta\hat{\sigma}_C(E_{\text{O-Ps}},p)\,\Psi(p) \tag{13}$$

where

$$\delta\hat{\sigma}_C(E_{\text{O-Ps}},p) \;\equiv\; \int \frac{d^3q}{(2\pi)^3}\,\delta\hat{\sigma}_0(|\mathbf{p}+\mathbf{q}|)\,\frac{1}{E_{\text{O-Ps}}-(\mathbf{p}+\mathbf{q})^2/m_e}\,V_C(q) \tag{14}$$

is a one-loop radiative correction to the basic decay.

To avoid double-counting the relativistic radiative corrections, we subtract the threshold value of $\delta\hat{\sigma}_C$, *i.e.*

$$\delta\hat{\sigma}_C(E,p) \to \delta\hat{\sigma}_C(E,p) - \delta\hat{\sigma}_C(0,0). \tag{15}$$

Thus the final contribution to the decay rate from $\delta\hat{\sigma}_0(p)$ is

$$\delta\Gamma_2 \;=\; 2\Psi(0)\int \frac{d^3p}{(2\pi)^3}\,\big(\delta\hat{\sigma}_C(E_{\text{O-Ps}},p) - \delta\hat{\sigma}_C(0,0)\big)\,\Psi(p). \tag{16}$$

Since $\delta\hat{\sigma}_0(p) \propto p^2$, this result simplifies to

$$\begin{aligned} \delta\Gamma_2 \;&=\; -\frac{\mathrm{E}_{\text{O-Ps}}}{m_e}\,\frac{19\pi^2-132}{12(\pi^2-9)}\,\hat{\sigma}_0(0) \\ &\approx\; 5\times 10^{-4}\mu\mathrm{s}^{-1}. \end{aligned} \tag{17}$$

Note that our original expression for this contribution (Eq.[11]) is ultraviolet divergent. The divergence comes from the high-$p$ region, which is precisely the region that must be removed so as to avoid double counting the relativistic radiative correction. Consequently, our subtraction regulates the integral. Such infinities are common when nonrelativistic expansions like that for $\hat{\sigma}_0(p)$ (Eq.[9]) are employed. However, they can always be removed by appropriate counterterms. The structure of these counterterms is readily determined using NRQED. The use of this theory puts calculations such as ours on a rigorous foundation, permitting systematic improvement of the final results (a rigorous derivation of the NRQED expansion to second order in $p/m$ and $\alpha$, including all counterterms, will be derived in another paper [5]).

3) CORRECTIONS TO $\Psi(0)$ : The wavefunction at the origin in the lowest-order rate (Eq.[3]) is modified by the various relativistic corrections that must be added to the Coulomb-Schrödinger theory. Using first-order perturbation theory, the correction is

$$\delta\Psi(0) \;=\; \sum_{n\neq\text{O-Ps}} \Psi_n(0)\frac{\langle n|\,\delta V\,|\text{O-Ps}\rangle}{E_{\text{O-Ps}}-E_n} \tag{18}$$

where the sum is over all eigenstates of the unperturbed theory, and

$$\begin{aligned}\delta V(\mathbf{p},\mathbf{q}) \quad = \quad & -(2\pi)^3\delta(\mathbf{p}-\mathbf{q})\,\frac{p^4}{4m_e^3} - \frac{e^2}{m_e^2 q^2}\,\big(p^2 - \frac{(\mathbf{p}\cdot\mathbf{q})^2}{q^2}\big) \\ & + \frac{3e^2}{4m_e^2} - \frac{e^2}{4m_e^2}\,\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + \frac{e^2}{4m_e^2}\,\frac{(\mathbf{q}\cdot\boldsymbol{\sigma}_1)\,(\mathbf{q}\cdot\boldsymbol{\sigma}_2)}{q^2} \\ & -3\frac{\alpha^2}{m_e^2}\ln(q^2/m_e^2). \end{aligned} \tag{19}$$

The potential $\delta V$ contains all $\mathcal{O}(v^2/c^2)$ corrections to the Schrödinger theory as well as the leading $\mathcal{O}(\alpha\, v^2/c^2)$ correction, given by the $\ln(q^2/m_e^2)$ term. This last term gives the leading contribution to the Lamb shift in positronium.

The corrections to the wavefunction will shift the decay rate by

$$\delta\Gamma_3(\text{O-Ps} \to 3\gamma) \;=\; 2\,|\delta\Psi(0)\,\Psi(0)|\;\hat{\sigma}_0(p). \tag{20}$$

As in the case of $\delta\hat{\sigma}(p)$, the ultraviolet divergences encountered in evaluating $\delta\Psi(0)$ are systematically removed by using NRQED. The remaining contributions all come from nonrelativistic loop momenta. We find

$$\delta\Psi(0) \;=\; \big[\frac{1}{6}\alpha^2\ln\alpha + 0.58\alpha^2 - \frac{3}{4\pi}\alpha^3(\ln\alpha)^2\big]\;\Psi(0) \tag{21}$$

which implies

$$\delta\Gamma_3 \;=\; \big[\frac{1}{3}\alpha^2\ln\alpha + 1.16\,\alpha^2 - \frac{3}{2\pi}\alpha^3(\ln\alpha)^2\big]\;\Gamma_0(\text{O-Ps}\to 3\gamma). \tag{22}$$

The logarithmic terms agree with the literature [3, 4]; the other correction is new.

By combining the $\mathcal{O}(\alpha^2\Gamma_0)$ contributions from all three of our corrections, we obtain the final result

$$\begin{aligned}\delta\Gamma_1 + \delta\Gamma_2 + \delta\Gamma_3|_{\alpha^2} \quad &= \quad \left\{\frac{19\pi^2 - 132}{12(\pi^2-9)}\,\frac{\alpha^2}{4} + 1.16\,\alpha^2 + c_2\left(\frac{\alpha}{\pi}\right)^2\right\}\Gamma_0 \\ &= \quad 9.56\times 10^{-4}\,\mu\text{s}^{-1} + c_2\;0.39\times 10^{-4}\,\mu\text{s}^{-1}. \end{aligned} \tag{23}$$

The coefficient $c_2$ is specified by the threshold rate for $e\overline{e} \to 3\gamma$ (Eqs.[6, 7, 8]). The known part of $c_2$ [7] contributes $11\times 10^{-4}\mu\text{s}^{-1}$ to the rate.

In this paper, we have outlined a new procedure for analyzing the O-Ps decay rate at $\mathcal{O}(\alpha^2\Gamma_0)$. We have computed all corrections in this order that depend in detail on bound-state physics. The only remaining contribution can be extracted from a calculation of the annihilation rate for a free electron and positron – no bound-state physics is required. Our corrections combined with the known part of the $c_2$ correction account for 20% of the difference between theory and experiment. The unexpectedly large size of these corrections makes

it plausible that the full calculation of $c_2$ will bring theory into agreement with experiment.

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